Mathematical Physics
(Individual Contest)

Prob. 1 The Hellmann-Feynman theorem: Given a Hamiltonian H which has discrete energy levels and smoothly depends on a coupling parameter γ, define $\mathcal{A} = \frac{\partial H}{\partial \gamma}$.

- For every eigenstate $|\Psi_j\rangle$ of H with its energy eigenvalue E_j, prove that $\langle \Psi_j|\mathcal{A}|\Psi_j\rangle = \frac{\partial E_j}{\partial \gamma}$.
- Taking as an example the one-dimensional quantum oscillator described by the Hamiltonian $H = -\frac{\partial^2}{\partial x^2} + \frac{\omega^2 x^2}{4}$, $\quad \omega \in R \quad$ and $\omega > 0$, show that

$$\langle \Psi_j|x^2|\Psi_j\rangle = \frac{(2j+1)}{\omega}.$$

Prob. 2 You are given the weak-field Newtonian limit of space-time as

$$ds^2 = -(1+2\phi)dt^2 + dx^2 + dy^2 + dz^2, \tag{1}$$

and the Newtonian gravitational potential ϕ satisfying $\vec{\nabla}^2\phi = 4\pi G\rho$ (i.e., $\vec{\nabla}^2 g_{00} = -8\pi G\rho$) with $\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$, G the Newton constant and ρ the matter density. Based on this information and the principle of general covariance, derive the complete Einstein field equation. You are given

$$\begin{aligned} R^\rho{}_{\mu\sigma\nu} &= \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma\Gamma - \text{terms} \\ \Gamma^\rho_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \end{aligned} \tag{2}$$

Prob.3 Consider a 2-form field in 6-dimensional spacetime

$$B_{(2)} = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu, \qquad \mu,\nu = 0,1,\cdots,5. \tag{3}$$

The free theory action is

$$S = -\int \mathrm{d}^6 x \left(\frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} + \frac{1}{4} m^2 B^{\mu\nu} B_{\mu\nu}\right), \tag{4}$$

where H is the corresponding 3-form field strength associated with the 2-form $B_{(2)}$

$$H_{(3)} = \frac{1}{6} H_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho = \frac{1}{2} \partial_{[\mu} B_{\nu\rho]} dx^\mu \wedge dx^\nu \wedge dx^\rho = dB_{(2)}\,. \tag{5}$$

(a)In the massive case, count the on-shell propagating degrees of freedom and describe the corresponding representation of Poincaré group.
(b)In the massless case, count the physical on-shell propagating degrees of freedom by eliminating the gauge redundancies, and describe the corresponding representation of Poincaré group.